USE OF ADJOINT FUNCTIONS IN INVESTIGATIONS OF HEAT CONDUCTION AND TRANSFER PROCESSES

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An examination is made of the use of adjoint functions in heat conduction and convection theory. Formulas of perturbation theory are obtained for steady and unsteady cases, an interpretation of the physical meaning of adjoint temperature is given, and some applications of the theory are discussed.

The use of adjoint functions and importance functions in neutron transport theory [1–3] has proved very fruitful because of formulas of the theory of perturbations and the possibility of more general treatment of variational problems concerning the optimum distribution of materials in media where there is radiation transfer [4]. The possibility is discussed below of using the technique of adjoint functions in the theory of heat transfer by means of conduction and convection.

1. We will examine the case of a steady temperature distribution in a heat-release element (HRE), cooled by a heat carrier at fixed temperature. The process is described by the heat conduction equation [5]

$$-\operatorname{div}(\lambda \vec{\nabla} t) = q_V \tag{1}$$

with the Newton boundary condition at the outside boundary of the HRE

$$-\lambda \nabla_n t\,|_{r_S} = \alpha t\,|_{r_S}. \tag{2}$$

Inside the HRE conditions are symmetrical and the temperature is finite, the latter being measured from the temperature of the carrier.

We formally write the equation [6] adjoint to (1):

$$-\operatorname{div}(\lambda \vec{\nabla} t^*) = p. \tag{3}$$

We call the function t*(r) the adjoint temperature and explain its physical meaning and that of p(r) below. It is not difficult to verify that the left sides of (1) and (3) are adjoint, if for t*(r) there occurs the boundary condition

$$-\lambda \nabla_n t^*\,|_{r_S} = \alpha t^*\,|_{r_S}. \tag{4}$$

In fact, multiplying (1) and (3) by t*(r) and t(r), respectively, subtracting the equations obtained one from another, and integrating over the whole volume of the HRE, we obtain

$$\int_V [-t^* \operatorname{div}(\lambda \vec{\nabla} t) + t \operatorname{div}(\lambda \vec{\nabla} t^*)]\, dV = \int_V q_V t^* dV - \int_V pt dV. \tag{5}$$

Transforming the left side of this equation with the aid of the relation $\operatorname{div}(\varphi \mathbf{A}) = \varphi \operatorname{div} \mathbf{A} + (\mathbf{A}, \vec{\nabla}\varphi)$, and using the Gauss theorem, we find

$$\int_V [-t^* \operatorname{div}(\lambda \vec{\nabla} t) + t \operatorname{div}(\lambda \vec{\nabla} t^*)]\, dV = \oint_S (-t^* \lambda \nabla_n t + t \lambda \nabla_n t^*)\, dS = \oint_S tt^* \left(-\frac{\lambda \nabla_n t}{t} + \frac{\lambda \nabla_n t^*}{t^*}\right) dS = 0$$

when conditions (2) and (4) are fulfilled. Therefore,

$$\int_V q_V t^* dV = \int_V pt dV = I, \tag{6}$$

and we may construct a theory of perturbations for the functional I. To this end we assume that in the HRE there has been an arbitrary perturbation of all the parameters

$$\Delta\lambda(\mathbf{r}) = \lambda'(\mathbf{r}) - \lambda(\mathbf{r}), \quad \Delta q_V(\mathbf{r}) = q'_V(\mathbf{r}) - q_V(\mathbf{r}),$$
$$\Delta\alpha(\mathbf{r}_S) = \alpha'(\mathbf{r}_S) - \alpha(\mathbf{r}_S), \quad \Delta p(\mathbf{r}) = p'(\mathbf{r}) - p(\mathbf{r}), \tag{7}$$

in such a way that the temperature has changed from t(**r**) to t'(**r**). Writing down the "perturbed" Eqs. (1) and (2) along with the conjugates (3) and (4), in which only the parameter p(**r**) is perturbed, and carrying out a cross multiplication of the equations by t*(**r**) and t'(**r**), a subtraction and an integration, we find the desired expression for the variation of the functional after simple transformations:

$$\Delta I = \int_V (p't' - pt)\, dV = \int_V \Delta q_V t^* dV - \int_V \Delta\lambda\, (\vec{\nabla} t',\ \vec{\nabla} t^*)\, dV - \oint_S \Delta\alpha\, t' t^* dS. \tag{8}$$

We note that in a number of practical cases it is convenient to use the formula for ΔI/I', since the linear-fractional functional is less sensitive to inaccuracies in the quantity t'(**r**).

We will analyze the physical meaning of the adjoint temperature t*(**r**). We assume that the Green's function $\Theta^*(\mathbf{r};\mathbf{r}_0)$ has been found for the adjoint equation, i.e., the solution of (3) and (4) under the assumption

$$p(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0). \tag{9}$$

Then in the more general case we have

$$t^*(\mathbf{r}) = \int_V \Theta^*(\mathbf{r};\ \mathbf{r}_0)\, p(\mathbf{r}_0)\, dV_0. \tag{10}$$

Analogously, if the Green's function $\Theta(\mathbf{r};\mathbf{r}_0)$ has been found for (1) and (2), then in the general case

$$t(\mathbf{r}) = \int_V \Theta(\mathbf{r};\ \mathbf{r}_0)\, q_V(\mathbf{r}_0)\, dV_0. \tag{11}$$

Substituting (10) and (11) into (6), we obtain, after changing the order of integration

$$\int_V p(\mathbf{r}_0)\,dV_0 \int_V \Theta^*(\mathbf{r};\ \mathbf{r}_0)\,q_V(\mathbf{r})\,dV = \int_V p(\mathbf{r})\,dV \int_V \Theta(\mathbf{r};\ \mathbf{r}_0)\,q_V(\mathbf{r}_0)\,dV_0. \quad (12)$$

From this relation, after replacing the variable of integration $\mathbf{r}_0$ by $\mathbf{r}$, we obtain the reciprocity theorem for the Green's functions

$$\Theta^*(\mathbf{r};\ \mathbf{r}_0) = \Theta(\mathbf{r}_0;\ \mathbf{r}). \quad (13)$$

The analogous reciprocity theorem for differential equations of the second order is well known in mathematics [6] and was proved in [2] for the kinetic equation of radiative transfer.

If follows from (13) that in the case of the action of a single heat source and when $p(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_0)$, the adjoint temperature at a given point $\mathbf{r}$ is the temperature at the point $\mathbf{r}_0$, if the heat source is shifted from the point $\mathbf{r}_0$ to the point $\mathbf{r}$. In the more general case of the value of the parameter $p(\mathbf{r})$ we obtain, with the aid of (10) and (13), the following relation:

$$t^*(\mathbf{r}) = \int_V \Theta(\mathbf{r}_0;\ \mathbf{r})\,p(\mathbf{r}_0)\,dV_0. \quad (14)$$

Thus, the adjoint temperature $t^*(\mathbf{r})$ is some linear functional of the temperature arising from the action of a source of unit power and depending on the coordinates of the location of this source. By analogy with the terminology used in neutron physics, the function $t^*(\mathbf{r})$ may also be called the importance function of the heat source with respect to the functional I.

We note that in the special case of steady heat conduction in a motionless medium, the differential equations and the boundary conditions for the Green's functions $\Theta(\mathbf{r};\mathbf{r}_0)$ and $\Theta^*(\mathbf{r};\mathbf{r}_0)$ are identical in form. This means that the solution of the two equations is identical, i.e.,

$$\Theta(\mathbf{r};\ \mathbf{r}_0) = \Theta^*(\mathbf{r};\ \mathbf{r}_0). \quad (15)$$

Comparing (13) and (15), we find an important temperature reversibility relation for the case examined:

$$\Theta(\mathbf{r};\ \mathbf{r}_0) = \Theta(\mathbf{r}_0;\ \mathbf{r}). \quad (16)$$

We will point out some other special cases of the functional I. If $p(\mathbf{r}) = \alpha\delta(\mathbf{r} - \mathbf{r}_S)$, functional I is equal to the heat flux in the HRE at the point $\mathbf{r}_S$ on its surface. In the case p = const the functional I is proportional to the mean integral temperature of the HRE.

3. We will examine a more general case—an unsteady problem of cooling of a HRE by a heat carrier flowing in a channel. The process of heat transfer by means of conduction and convection in this system is described by the equation [5]

$$C\gamma\frac{\partial t}{\partial \tau} + C\gamma(\mathbf{W}, \vec{\nabla} t) - \operatorname{div}(\lambda\vec{\nabla} t) = q_V. \quad (17)$$

The parameters appearing in (17) are piecewise-continuous functions of the coordinates. The boundary conditions of the problem are the requirements of continuity of temperature $t(\mathbf{r}, \tau)$ and of heat flux $-\lambda\nabla_n t$ at the interface between the HRE and the heat carrier, the finiteness of the condition of symmetry of temperature, and the condition of Newtonian heat loss at the outer surface of the channel $-\lambda\nabla_n t|_{S_{ch}} = \alpha t|_{S_{ch}}$. The initial condition may always be represented in the form

$$t(\mathbf{r},\ -\infty) = 0. \quad (18)$$

The equation adjoint to (17) has the form

$$-C\gamma\frac{\partial t^*}{\partial \tau} - C\gamma(\mathbf{W}, \vec{\nabla} t^*) - \operatorname{div}(\lambda\vec{\nabla} t^*) = p. \quad (19)$$

The boundary conditions for the adjoint temperature $t^*(\mathbf{r}, \tau)$ are the same as for $t(\mathbf{r}, \tau)$, and the adjoint initial condition has the form

$$t^*(\mathbf{r},\ \infty) = 0. \quad (20)$$

If a certain space-time perturbation of all the parameters takes place in the system, then the formula of the theory of perturbations for the functional

$$I = \int_{-\infty}^{\infty}\int_V \frac{p}{C\gamma}\,t\,dV d\tau = \int_{-\infty}^{\infty}\int_V \frac{q_V}{C\gamma}\,t^* dV d\tau, \quad (21)$$

as may easily be shown, may be written in the following form:

$$\Delta I = \int_{-\infty}^{\infty}\int_V\left(\frac{p'}{C\gamma}t' - \frac{p}{C\gamma}t\right)dV d\tau = \int_{-\infty}^{\infty}\int_V t't^*\,\Delta\left(\frac{q_V}{C\gamma}\right)dV d\tau + \int_{-\infty}^{\infty}\int_V t'(\Delta\mathbf{W}', \vec{\nabla} t^*)\,dV d\tau + \int_{-\infty}^{\infty}\int_V t't^*\operatorname{div}\mathbf{W}'\,dV d\tau -$$
$$- \int_{-\infty}^{\infty}\int_V\left[\lambda'\left(\vec{\nabla} t', \vec{\nabla}\frac{t^*}{C'\gamma'}\right) - \lambda\left(\vec{\nabla} t^*, \vec{\nabla}\frac{t'}{C\gamma}\right)\right]dV d\tau -$$
$$- \int_{-\infty}^{\infty}\int_{S_K} t't^*\,\Delta\left(\frac{\alpha}{C\gamma}\right)dS d\tau - \int_{-\infty}^{\infty} d\tau\left[\int_{F_{out}} t^*t'W_n'dS - \int_{F_{in}} t^*t'W_n'dS\right]. \quad (22)$$

In deriving (22) we used the Gauss theory and omitted the term $\int_{-\infty}^{\infty}\oint_{S_K} t't^*W_n'dS d\tau$ because the normal component of flow velocity at the wall of the channel is zero.

We will point out some special cases of the functional (21). If $p(\mathbf{r}, \tau) = C\gamma\delta(\mathbf{r} - \mathbf{r}_0)\,\delta(\tau - \tau_0)$, then the functional of perturbation theory is the temperature $t(\mathbf{r}_0, \tau_0)$. If we assume that $p(\mathbf{r}, \tau) = C\gamma\alpha\delta(\mathbf{r} - \mathbf{r}_S)\times \delta(\tau - \tau_0)$, then the functional of the problem becomes the heat flux at the point $\mathbf{r}_S$ of the surface at time τ_0. We may also make the functional of the problem the heat content of the stream (local, averaged over the section or over the volume), for which we must introduce the velocity distribution in the expression $p(\mathbf{r}, \tau)$. Similarly, we may, in the case examined, prove the reciprocity theorem for the Green's functions

$$\Theta(\mathbf{r},\ \tau;\ \mathbf{r}_0,\ \tau_0) = \Theta^*(\mathbf{r}_0,\ \tau_0;\ \mathbf{r},\ \tau) \quad (23)$$

and interpret the physical meaning of the adjoint temperature with the help of the relation

$$t^*(\mathbf{r}, \tau) = = \int_{-\infty}^{\infty} \int_V \frac{p(\mathbf{r}_0, \tau_0)}{C(r_0, \tau_0)\gamma(r_0, \tau_0)} \Theta(\mathbf{r}_0, \tau_0; \mathbf{r}, \tau)\, dV_0 d\tau_0. \quad (24)$$

It is evident that the condition of temperature reversibility, analogous to (16), does not hold in the general case.

4. We will discuss some examples of the use of perturbation theory. Formulas (8) and (22) make it possible, using the unperturbed functions that have been found, $t(\mathbf{r}, \tau)$ and $t^*(\mathbf{r}, \tau)$, to find the change in the value of I with change in the parameters of the problem, in the first approximation. This is especially important when direct solution of the problem is difficult, even for numerical calculation (for example, when the perturbation is local in nature) or the required accuracy cannot be obtained. In a number of cases, even if the perturbed problem is solved directly, a more accurate value of ΔI may be calculated from (8) or (22) by substituting t' and t*. Typical cases when it is useful to apply perturbation theory are cases of approximate solutions of problems in heat conduction theory on the basis of simplifying assumptions as to the nature of the physical constants. In these cases we may take an estimate of the error in the value of the functional of interest from the assumption made. Then we can develop the theory of high-order perturbations, which is especially suitable if the adjoint function is expressed analytically.

Formulas (8) and (22) are also useful for problems in which it is difficult to find a direct solution because of an angular dependence of the heat removal or the heat-generating sources. We will examine as an example the steady problem of cooling of an infinitely long cylindrical HRE with internal heat sources and heat loss according to Newton's law. Taking into account that the filament-shaped heat source is located at the point with coordinates r_0, φ_0, we will find the Green's function for the temperature in this problem. Assuming that λ and α do not depend on the coordinates, we obtain the following solution:

$$\Theta(r, \varphi; r_0, \varphi_0) = \Theta^*(r, \varphi; r_0, \varphi_0) = \frac{1}{2\pi\alpha R} + \frac{1}{2\pi\lambda}\sum_{k=1}^{\infty} \frac{\alpha R/\lambda}{k + \alpha R/\lambda}\left(\frac{1}{\alpha R/\lambda} - \frac{1}{k}\right)\left(\frac{r_0}{R}\right)^k \left(\frac{r}{R}\right)^k \times$$
$$\times \cos k(\varphi - \varphi_0) - \frac{1}{4\pi\lambda}\ln\left[\left(\frac{r}{R}\right)^2 + \left(\frac{r_0}{R}\right)^2 - 2\frac{r}{R}\frac{r_0}{R}\cos(\varphi - \varphi_0)\right]. \quad (25)$$

We note that (25) is symmetrical with respect to inversion of the source coordinates and the point of temperature observation [formula (16)]. We will consider the expression for t(r) with constant q_V, λ and α as the zero-order approximation

$$t_0(r) = q_V(R^2 - r^2)/4\lambda + q_V R/2\alpha. \quad (26)$$

Formula (8), which allows one to find a correction for the temperature in the first approximation at an arbitrary point (r_0, φ_0), due to q_V, λ and α not being constant, has the form

$$\Delta t_1(r_0, \varphi_0) = t_1'(r_0, \varphi_0) - t_0(r_0) = \int_0^R \int_0^{2\pi} \Delta q_V(r, \varphi)\Theta^*(r, \varphi; r_0, \varphi_0)\, r dr d\varphi -$$
$$- \frac{q_V}{2\lambda}\int_0^R \int_0^{2\pi} \Delta\lambda(r, \varphi)\, r \frac{\partial \Theta^*(r, \varphi; r_0, \varphi_0)}{\partial r}\, r dr d\varphi - \frac{q_V R^2}{2\alpha}\int_0^{2\pi} \Delta\alpha(\varphi)\Theta^*(R, \varphi; r_0\varphi_0)\, d\varphi. \quad (27)$$

From the value of $t_1'(r_0, \varphi_0)$ found we can improve the accuracy of $\Delta\lambda(r,\varphi)$ and $\Delta\alpha(\varphi)$, after which we find $\Delta t_2(r_0, \varphi_0)$, and so on. The correction in the (n + 1)-th approximation in terms of the temperature in the n-th approximation is

$$\Delta t_{n+1}(r_0, \varphi_0) = t_{n+1}'(r_0, \varphi_0) - t_0(r_0) = \int_0^R \int_0^{2\pi} \Delta q_V(r, \varphi)\Theta^*(r, \varphi; r_0, \varphi_0)\, r dr d\varphi -$$
$$- \int_0^R \int_0^{2\pi} \Delta\lambda(r, \varphi)\left[\frac{\partial t_n'(r, \varphi)}{\partial r}\frac{\partial \Theta^*(r, \varphi; r_0, \varphi_0)}{\partial r} + \frac{1}{r^2}\frac{\partial t_n'(r, \varphi)}{\partial \varphi} \times \frac{\partial \Theta^*(r, \varphi; r_0, \varphi_0)}{\partial \varphi}\right] r dr d\varphi -$$
$$- \int_0^{2\pi} \Delta\alpha(\varphi)\, t_n'(R, \varphi)\Theta^*(R, \varphi; r_0, \varphi_0)\, R d\varphi. \quad (28)$$

We will illustrate the convergence of the above method of calculating higher-order perturbations by an example amenable to exact examination. To this end we will examine the previous problem under the assumption that there is no angular dependence of any of the parameters. It is not difficult to find the Green function of this problem:

$$\Theta(r; r_0) = \Theta^*(r; r_0) = \begin{cases} \dfrac{1}{2\pi\lambda}\ln\dfrac{R}{r_0} + \dfrac{1}{2\pi\alpha R} & r \leqslant r_0 \\ \dfrac{1}{2\pi\lambda}\ln\dfrac{R}{r} + \dfrac{1}{2\pi\alpha R} & r \geqslant r_0. \end{cases} \quad (29)$$

We calculate from (8) the change in $t_0(r)$ due to the variable $\lambda'[t'(r)] = \lambda'(r)$:

$$\Delta t(r_0) = t'(r_0) - t_0(r_0) = -\int_0^R \Delta\lambda(r)\frac{dt'}{dr}\frac{\partial \Theta^*}{\partial r}\, 2\pi r dr, \quad (30)$$

where

$$\Delta\lambda(r) = \lambda'(r) - \lambda, \qquad \frac{\partial\Theta^*}{\partial r} = \begin{cases} 0 & r \leqslant r_0 \\ -1/2\pi\lambda r & r \geqslant r_0. \end{cases}$$

It can be shown that, for the problem examined,

$$\Delta t_1(r_0) = -\frac{q_V}{4\pi\lambda}\int_{r_0}^{R}\frac{\Delta\lambda(r)}{\lambda}2\pi r dr,$$

$$\Delta t_2(r_0) = -\frac{q_V}{4\pi\lambda}\int_{r_0}^{R}\frac{\Delta\lambda(r)}{\lambda}\left[1 - \frac{\Delta\lambda(r)}{\lambda}\right]2\pi r dr,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (31)$$

$$\Delta t_n(r_0) = -\frac{q_V}{4\pi\lambda}\int_{r_0}^{R}\frac{\Delta\lambda(r)}{\lambda}\sum_{k=0}^{n-1}\left[-\frac{\Delta\lambda(r)}{\lambda}\right]^k 2\pi r dr =$$

$$= -\frac{q_V}{4\pi\lambda}\int_{r_0}^{R}\frac{\Delta\lambda(r)}{\lambda'(r)}\left\{1 - \left[-\frac{\Delta\lambda(r)}{\lambda}\right]^n\right\}2\pi r dr.$$

In the last relation use was made of the formula for the sum of terms of a geometric progression. If the ratio of this progression $\left|\frac{\Delta\lambda}{\lambda}\right| < 1$, then as $n \to \infty$ we have

$$\Delta t_\infty(r_0) = -\frac{q_V}{4\pi\lambda}\int_{r_0}^{R}\frac{\lambda'(r) - \lambda}{\lambda'(r)}2\pi r dr. \quad (32)$$

It is not difficult to verify that this passage to the limit gives an exact relation. For this we solve the exact heat conduction equation

$$\lambda'(r)\frac{d}{dr}\left(r\frac{dt'}{dr}\right) + \frac{d\lambda'}{dr}r\frac{dt'}{dr} = -q_V r. \quad (33)$$

Taking account of the conditions

$$\left.\frac{dt'}{dr}\right|_{r=0} = 0, \qquad -\lambda'\left.\frac{dt'}{dr}\right|_R = \alpha t'\Big|_R,$$

we obtain

$$t'(r) = \frac{q_V R}{2\alpha} + \frac{q_V}{2\lambda}\int_r^R\frac{\lambda}{\lambda'(r)}r'dr'. \quad (34)$$

Using this relation to determine the temperature t(r) corresponding to the case $\lambda' = \lambda = \text{const}$, we find

$$\Delta t(r) = t'(r) - t(r) =$$

$$= -\frac{q_V}{4\pi\lambda}\int_r^R\frac{\lambda'(r') - \lambda}{\lambda'(r')}2\pi r'dr'. \quad (35)$$

It is seen that expressions (32) and (35) coincide identically.

5. We will enumerate some other cases where it is useful to apply perturbation theory. Formula (22) permits us to simplify the problem of finding the unsteady temperature field in a HRE with a shell and process layers, since the effect of the latter may be considered to be a local perturbation of the quantities λ, $C\gamma$ and q_V.

Perturbation theory allows us to calculate correctly the effect of various tolerances and deviations from nominal (inaccuracy and scatter in the thermophysical constants, in the heat release sources, in the heat transfer coefficient, in the thicknesses of the materials, etc.) on the temperature of the HRE or on the heat flux at a dangerous point.

Formulas (8) and (22) may undoubtedly be of advantage to the experimenter. For example, in calculating the true temperature of the wall of a working section according to the readings of thermocouples embedded within the wall, he may evaluate the influence of local variation of λ or of q_V in the places where the thermocouples have been embedded on the local variation of the heat flux.

NOTATION

$\lambda(r,\tau)$ is the thermal conductivity; $t(r,\tau)$ is the temperature; $t^*(r,\tau)$ is the adjoint temperature; $q_V(r,\tau)$ is the density of heat release sources; $p(r,\tau)$ is a parameter of adjoint equation; r is the generalized coordinate; τ is time; $\alpha(\mathbf{r}_S,\tau)$ is the heat transfer coefficient; I is the linear functional of temperature; $\Theta(\mathbf{r},\tau;\mathbf{r}_0,\tau_0)$ and $\Theta^*(\mathbf{r},\tau;\mathbf{r}_0,\tau_0)$ is the Green's function for $t(\mathbf{r},\tau)$ and $t^*(\mathbf{r},\tau)$; $C\gamma(\mathbf{r},\tau)$ is the volume specific heat; $W(\mathbf{r},\tau)$ is the vector distribution of flow velocities; V, S are the volume and surface areas of body; R is the radius of HRE; r, φ are the radial and angular coordinates; F_{in}, F_{out} are the inlet and outlet flow areas of channel.

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